AN ALGORITHM TO FIND OPTIMUM COST TIME TRADE OFF PAIRS IN A FRACTIONAL CAPACITATED TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

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Abstract:

This paper presents an algorithm to find optimum cost - time trade off pairs in a fractional capacitated transportation problem with bounds on total availabilities at sources and total destination requirements. The objective function is a ratio of two linear functions consisting of variable costs and profits respectively. Sometimes, situations arise where either reserve stocks have to be kept at the supply points say, for emergencies or there is a shortfall in the production level. In such situations, the total flow needs to be curtailed. In this paper, a special class of transportation problems is studied where in the total transportation flow is restricted to a known specified level. A related transportation problem is formulated and the efficient cost- time trade off pairs to the given problem are shown to be derivable from this related transportation problem. Moreover , it is established that special type of feasible solution called corner feasible solution of related transportation problem bear one to one correspondence with the feasible solution of the given restricted flow problem . The optimal solution to restricted flow problem may be obtained from the optimal solution to related transportation problem. Numerical illustration is included in support of theory.

Keywords: Capacitated transportation problem, restricted flow, fixed charge, related problem, corner feasible solution, trade off pairs.

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1 Introduction:

There is a wide scope of capacitated transportation problem with bounds on rim conditions. It can be used extensively in telecommunication networks, production – distribution systems, rail and urban road system when there is a limited capacity of resources such as vehicles, docks, equipment capacity etc. These are bounded variable transportation problems. Many researchers like Dahiya and Verma [3], Misra and Das [8] have contributed in this field.

Another class of transportation problem is a non linear programming problem where the objective function to be optimized is a ratio of two linear functions. Optimization of a ratio of criteria often describes some kind of an efficiency measure for a system . Fractional programs finds its application in a variety of real world problems such as stock cutting problem , resource allocation problems , routing problem for ships and planes , cargo – loading problem , inventory problem and many other problems. Dahiya and Verma [4] studied paradox in a non linear capacitated transportation problem. Arora et .al [7] studied indefinite quadratic transportation problems. Khurana et. al.[5] studied restricted and enhanced flow in the sum of a linear and linear fractional transportation problem in 2006. Verma and Puri [10] studied paradox in a linear fractional transportation problem in 1991.

In 1994, Basu et.al. [2] developed an algorithm for the optimum cost- time trade off pairs in a fixed charge linear transportation problem giving same priority to cost as well as time. In 2004, Arora et.al.[1]also studied time cost trade off pairs in a three dimensional fixed charge indefinite quadratic transportation problem.

Many researchers like Arora [6], Thirwani [9] have studied restricted flow problems. Sometimes, situations arise when reserve stocks are to be kept at sources for emergencies . This gives rise to restricted flow problem where the total flow is restricted to a known specified level. This motivated us to develop an algorithm to find the optimum cost - time trade off pairs in a fractional capacitated transportation problem with restricted flow.

2 Problem Formulations:

Consider a fractional capacitated transportation problem given by

$$(P1):\ \min\left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}},\ \max_{i\in I, j\in J}\ t_{ij}/x_{ij}>0\right]$$

subject to

$$a_i \le \sum_{j\in J} x_{ij} \le A_i\ ;\ \forall\, i\in I \qquad (1)$$

$$b_j \le \sum_{i\in I} x_{ij} \le B_j\ ;\ \forall\ j\in J \qquad (2)$$

$$l_{ij} \le x_{ij} \le u_{ij}\ \text{ and integers }\ \forall\, i\in I,\, j\in J \qquad (3)$$

$$\sum_{i\in I}\sum_{j\in J} x_{ij} = P\left(<\min\left(\sum_{i\in I} A_i, \sum_{j\in J} B_j\right)\right) \qquad (4)$$

$I = \{1, 2, \ldots m\}$ is the index set of m origins.

$J = \{1, 2, \ldots, n\}$ is the index set of n destinations.

x_{ij} = number of units transported from origin i to the destination j.

c_{ij} = per unit pilferage cost when shipment is sent from i^{th} origin to the j^{th} destination.

d_{ij} = the variable profit per unit amount transported from i^{th} origin to the j^{th} destination.

l_{ij} and u_{ij} are the bounds on number of units to be transported from i^{th} origin to j^{th} destination.

a_i and A_i are the bounds on the availability at the i^{th} origin, i I

b_j and B_j are the bounds on the demand at the j^{th} destination, j J

t_{ij} is the time of transporting goods from i^{th} origin to the j^{th} destination.

It is assumed that $\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij} > 0$ for every feasible solution X satisfying (1),(2),(3) and (4) and all upper bounds u_{ij} ; $(i,j) \in I\times J$ are finite.

Sometimes, situations arise when one wishes to keep reserve stocks at the origins for emergencies, there by restricting the total transportation flow to a known specified level, say P $\left(<\min\left(\sum_{i\in I} A_i, \sum_{j\in J} B_j\right)\right)$.This flow constraint in the problem (P1) implies that a total

$\left(\sum_{i\in I} A_i - P\right)$ of the source reserves has to be kept at the various sources and a total $\left(\sum_{j\in J} B_j - P\right)$ of destination slacks is to be retained at the various destinations. Therefore an extra destination to receive the source reserves and an extra source to fill up the destination slacks are introduced.

In order to solve the problem (P1), we separate it in to two problems (P2) and (P3) where

(P2): minimize the cost function $\left[\dfrac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}}\right]$ subject to (1), (2), (3) and (4).

(P3): minimize the time function $\max_{i\in I, j\in J} \; t_{ij} / x_{ij} > 0$ subject to (1), (2), (3) and (4).

In order to solve the problem (P2) we convert it in to related problem (P2´) given below.

$$(P2'):\ \min z = \left[\frac{\sum_{i\in I'}\sum_{j\in J'} c'_{ij}y_{ij}}{\sum_{i\in I'}\sum_{j\in J'} d'_{ij}y_{ij}}\right]$$

subject to

$$\sum_{j\in J'} y_{ij} = A'_i \qquad \forall i \in I' \qquad (5)$$

$$\sum_{i\in I'} y_{ij} = B'_j \qquad \forall j \in J' \qquad (6)$$

$$l_{ij} \le y_{ij} \le u_{ij} \qquad \forall\ i, j\ \in I \times J \qquad (7)$$

$$0 \le y_{m+1,j} \le B_j - b_j \qquad \forall j \in J$$

$$0 \le y_{i,n+1} \le A_i - a_i \qquad \forall i \in I$$

$$y_{m+1,n+1} = 0$$

$$A_i' = A_i\ \forall i\in I, \quad A'_{m+1} = \sum_{j\in J} B_j - P\ ,\ B'_j = B_j\ \forall j\in J\ ,\ B'_{n+1} = \sum_{i\in I} A_i - P$$

$$c'_{ij} = c_{ij}\ ,\ \forall i\in I, j\in J,\ \ c'_{m+1,j} = c'_{i,n+1} = 0 \quad \forall i\in I,\ \ \forall j\in J,\ \ c'_{m+1,n+1} = M$$

$d'_{ij} = d_{ij} \;\; \forall i \in I, j \in J, \;\; d'_{m+1,j} = d'_{i,n+1} = 0 \quad \forall i \in I, \;\; \forall j \in J; \;\; d'_{m+1,n+1} = M$ where M is a large positive number. $I' = \{1, 2, \ldots m, m+1\}$, $J' = \{1, 2, \ldots n, n+1\}$

In order to solve the problem (P3), we convert it to the related problem (P3´) given below.

(P3´): $\min T = \max \; t'_{ij} / y_{ij} > 0 \quad \forall i \in I'$ and $\forall j \in J'$ subject to

$$\sum_{j \in J'} y_{ij} = A_i' \quad \forall i \in I'$$

$$\sum_{i \in I'} y_{ij} = B_j' \quad \forall j \in J'$$

$$l_{ij} \le y_{ij} \le u_{ij} \quad \forall \; i, j \; \in I \times J$$

$$0 \le y_{m+1,j} \le B_j - b_j \quad \forall j \in J$$

$$0 \le y_{i,n+1} \le A_i - a_i \quad \forall i \in I$$

$$y_{m+1,n+1} = 0$$

$$A_i' = A_i \;\; \forall i \in I, \quad A'_{m+1} = \sum_{j \in J} B_j - P \;\; , B'_j = B_j \;\; \forall j \in J \quad , B'_{n+1} = \sum_{i \in I} A_i - P$$

$$c'_{ij} = c_{ij} \;\; , \forall i \in I, j \in J, \;\; c'_{m+1,j} = c'_{i,n+1} = 0 \quad \forall i \in I, \;\; \forall j \in J, \;\; c'_{m+1,n+1} = M$$

$$d'_{ij} = d_{ij} \;\; \forall i \in I, j \in J, \; d'_{m+1,j} = d'_{i,n+1} = 0 \quad \forall i \in I, \;\; \forall j \in J \;\; d'_{m+1,n+1} = M$$

$$t'_{ij} = t_{ij} \;\; \forall \; i, j \; \in I \times J \; , t'_{m+1,j} = t'_{i,n+1} = 0 \; \forall i \in I, j \in J$$

$$t'_{m+1,n+1} > \max \; t_{ij} / x_{ij} > 0 \;\; \forall i \in I, j \in J$$

To obtain the set of efficient cost - time trade off pairs, we first solve the problem (P2´) and read the time with respect to the minimum cost Z where time T is given by the problem (P3´)

At the first iteration, let Z_1^* be the minimum total cost of the problem (P2´) . Find all alternate solutions i.e. solutions having the same value of $Z = Z_1^*$. Let these solutions be $X_1, X_2, \ldots X_n$. Corresponding to these solutions, find the time $T_1^* = \min_{X_1, X_2 \ldots X_n} \; \max_{i \in I, j \in J'} \; t_{ij} / x_{ij} > 0$ .Then (Z_1^*, T_1^*) is called the first cost time trade off pair. Modify the cost with respect to the time so obtained i.e. define $c_{ij} = \begin{Bmatrix} M & \text{if } t_{ij} \geq T^* \\ c_{ij} & \text{if } t_{ij} < T^* \end{Bmatrix}$ and form the new problem (P2´´) and find its optimal solution and all feasible alternate solutions. Let the new value of Z be Z_2* and the corresponding time is T_2* , then (Z_2*, T_2*) is the second cost time trade off pair. Repeat this process. Suppose that after q th iteration ,the problem becomes infeasible. Thus, we get the following complete set of cost- time trade off pairs. (Z_1*, T_1*) , (Z_2*, T_2*),(Z_3*,T_3*),…………….(Z_q*, T_q*) where $Z_1^* \leq Z_2^* \leq Z_3^* \leq \ldots\ldots\ldots \leq Z_q^*$ and $T_1^* > T_2^* > T_3^* \ldots\ldots\ldots > T_q^*$.The pairs so obtained are pareto optimal solutions of the given problem. Then we identify the minimum cost Z_1* and minimum time T_q* among the above trade off pairs. The pair (Z_1*, T_q*) with minimum cost and minimum time is termed as the ideal pair which can not be achieved in practical situations.

3 Theoretical Development:

Theorem1: A feasible solution $X^0 = \{\overset{\circ}{x}_{ij}\}_{I \times J}$ of problem (P2) with objective function value $\frac{N^\circ}{D^\circ}$ will be a local optimum basic feasible solution iff the following conditions holds.

$$\delta_{ij}^1 = \frac{\theta_{ij}\left[D^\circ(c_{ij} - z_{ij}^1) - N^\circ(d_{ij} - z_{ij}^2)\right]}{D^\circ\left[D^\circ + \theta_{ij}(d_{ij} - z_{ij}^2)\right]} \geq 0; \forall (i,j) \in N_1$$

$$\delta_{ij}^2 = -\frac{\theta_{ij}\left[D^\circ(c_{ij} - z_{ij}^1) - N^\circ(d_{ij} - z_{ij}^2)\right]}{D^\circ\left[D^\circ - \theta_{ij}(d_{ij} - z_{ij}^2)\right]} \geq 0; \forall (i,j) \in N_2$$

and if X^0 is an optimal solution of (P2),then $\delta_{ij}^1 \geq 0; \forall (i,j) \in N_1$ and $\delta_{ij}^2 \geq 0; \forall (i,j) \in N_2$ where $N^\circ = \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}^\circ$, $D^\circ = \sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}^\circ$, B denotes the set of cells (i,j) which are basic and N_1 and N_2 denotes the set of non basic cells (i,j) which are at their lower bounds and upper bounds respectively.

$u_i^1, u_i^2, v_j^1, v_j^2; i \in I, j \in J$ are the dual variables such that $u_i^1 + v_j^1 = c_{ij}$, $\forall (i,j) \in B$; $u_i^2 + v_j^2 = d_{ij}$, $\forall (i,j) \in B$; $u_i^1 + v_j^1 = z_{ij}^1, \forall (i,j) \notin B$; $u_i^2 + v_j^2 = z_{ij}^2$, $\forall (i,j) \notin B$

Proof: Let $X^0 = \{\overset{\circ}{x}_{ij}\}_{I \times J}$ be a basic feasible solution of problem (P2) with equality constraints. Let z^0 be the corresponding value of objective function. Then

$$z^\circ = \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}^\circ}{\sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}^\circ}\right] = \frac{N^\circ}{D^\circ} \text{ (say)}$$

$$= \left[\frac{\sum_{i\in I}\sum_{j\in J}(c_{ij} - u_i^1 - v_j^1)x_{ij}^\circ + \sum_{i\in I}\sum_{j\in J}(u_i^1 + v_j^1)x_{ij}^\circ}{\sum_{i\in I}\sum_{j\in J}(d_{ij} - u_i^1 - v_j^1)x_{ij}^\circ + \sum_{i\in I}\sum_{j\in J}(u_i^2 + v_j^2)x_{ij}^\circ}\right]$$

$$= \left[\frac{\sum\sum_{(i,j)\in N_1}(c_{ij} - u_i^1 - v_j^1)l_{ij} + \sum\sum_{(i,j)\in N_2}(c_{ij} - u_i^1 - v_j^1)u_{ij} + \sum_{i\in I}\sum_{j\in J}(u_i^1 + v_j^1)x_{ij}^\circ}{\sum\sum_{(i,j)\in N_1}(d_{ij} - u_i^2 - v_j^2)l_{ij} + \sum\sum_{(i,j)\in N_2}(d_{ij} - u_i^2 - v_j^2)u_{ij} + \sum_{i\in I}\sum_{j\in J}(u_i^2 + v_j^2)x_{ij}^\circ}\right]$$

$$= \left[\frac{\sum\sum_{(i,j)\in N_1}(c_{ij} - z_{ij}^1)l_{ij} + \sum\sum_{(i,j)\in N_2}(c_{ij} - z_{ij}^1)u_{ij} + \sum_{i\in I}a_i u_i^1 + \sum_{j\in J}b_j v_j^1}{\sum\sum_{(i,j)\in N_1}(d_{ij} - z_{ij}^2)l_{ij} + \sum\sum_{(i,j)\in N_2}(d_{ij} - z_{ij}^2)u_{ij} + \sum_{i\in I}a_i u_i^2 + \sum_{j\in J}b_j v_j^2}\right]$$

Let some non basic variable $x_{ij} \in N_1$ undergoes change by an amount θ_{rs} where θ_{rs} is given by

$$\min\begin{Bmatrix} u_{rs} - l_{rs} \\ x_{ij}^\circ - l_{ij} \text{ for all basic cells (i, j) with a } (-\theta) \text{ entry in } \theta-\text{loop} \\ u_{ij} - x_{ij}^\circ \text{ for all basic cells (i, j) with a } (+\theta) \text{ entry in } \theta-\text{loop} \end{Bmatrix}$$

Then new value of the objective function $\hat{z}$ will be given by

$$\hat{z} = \frac{N^\circ + \theta_{rs}(c_{rs} - z_{rs}^1)}{D^\circ + \theta_{rs}(d_{rs} - z_{rs}^2)}$$

$$\hat{z} - z^\circ = \left[\frac{N^\circ + \theta_{rs}(c_{rs} - z_{rs}^1)}{D^\circ + \theta_{rs}(d_{rs} - z_{rs}^2)} - \frac{N^\circ}{D^\circ}\right]$$

$$= \frac{\theta_{rs}\left[D^\circ(c_{rs} - z_{rs}^1) - N^\circ(d_{rs} - z_{rs}^2)\right]}{D^\circ\left[D^\circ + \theta_{rs}(d_{rs} - z_{rs}^2)\right]} = \delta_{rs}^1 \text{ (say)}$$

Similarly, when some non basic variable $x_{pq} \in N_2$ undergoes change by an amount θ_{pq} then

$$\hat{z} - z^{\circ} = -\frac{\theta_{pq}\left[D^{\circ}(c_{pq} - z^{1}_{pq}) - N^{\circ}(d_{pq} - z^{2}_{pq})\right]}{D^{\circ}\left[D^{\circ} - \theta_{pq}(d_{pq} - z^{2}_{pq})\right]} = \delta^{2}_{pq}\,(\text{say})$$

Hence X^0 will be local optimal solution iff $\delta^1_{ij} \geq 0; \forall (i,j) \in N_1$ and $\delta^2_{ij} \geq 0; \forall (i,j) \in N_2$.If X^0 is a global optimal solution of (P2), then it is an optimal solution and hence the result follows.

Definition: Corner feasible solution : A basic feasible solution $\{y_{ij}\}$ $i \in I'$, $j \in J'$ to(P2´) is called a corner feasible solution (cfs) if $y_{m+1,n+1} = 0$

Theorem 2. A non corner feasible solution of (P2´) cannot provide a basic feasible solution to (P2).

Proof: Let $\{y_{ij}\}_{I' \times J'}$ be a non corner feasible solution to (P2´) .Then $y_{m+1,n+1} = \lambda$ (>0)

Thus $\sum_{i \in I'} y_{i,n+1} = \sum_{i \in I} y_{i,n+1} + y_{m+1,n+1}$

$$= \sum_{i \in I} y_{i,n+1} + \lambda$$

$$= \sum_{i \in I} A_i - P$$

Therefore, $\sum_{i \in I} y_{i,n+1} = \sum_{i \in I} A_i - (P + \lambda)$ (8)

Now, for $i \in I$,

$\sum_{j \in J'} y_{ij} = A'_i = A_i$

$\Rightarrow \sum_{i \in I} \sum_{j \in J'} y_{ij} = \sum_{i \in I} A_i$ (9)

(8) and (9) implies that $\sum_{i \in I} \sum_{j \in J} y_{ij} = P + \lambda$

This implies that total quantity transported from all the sources in I to all the destinations in J is P + λ > P, a contradiction to the assumption that total flow is P and hence $\{y_{ij}\}_{I' \times J'}$ cannot provide a feasible solution to (P2).

Lemma 1: There is a one –to-one correspondence between the feasible solution to (P2) and the corner feasible solution to (P2´).

Proof: Let $\{x_{ij}\}_{I\,xJ}$ be a feasible solution of (P2).So $\{x_{ij}\}_{I\,xJ}$ will satisfy (1) to (4).

Define $\{y_{ij}\}_{I'\,xJ'}$ by the following transformation

$y_{ij} = x_{ij}$, $i \in I,\ j \in J$

$y_{i,n+1} = A_i - \sum_{j \in J} x_{ij}$, $i \in I$

$y_{m+1,j} = B_j - \sum_{i \in I} x_{ij}$, $j \in J$

$y_{m+1,n+1} = 0$

It can be shown that $\{y_{ij}\}_{I'\,xJ'}$ so defined is a cfs to (P2´)

Relation (1) to (3) implies that

$l_{ij} \le y_{ij} \le u_{ij}$ for all $i \in I,\ j \in J$

$0 \le y_{i,\,n+1} \le A_i - a_i$, $i \in I$

$0 \le y_{m+1,\,j} \le B_j - b_j$, $j \in J$

$y_{m+1,n+1} \ge 0$,

Also for $i \in I$

$$\sum_{j \in J'} y_{ij} = \sum_{j \in J} y_{ij} + y_{i,\,n+1} = \sum_{j \in J} x_{ij} + A_i - \sum_{j \in J} x_{ij} = A_i = A_i^{'}$$

For i = m+1

$$\sum_{j \in J'} y_{m+1,\,j} = \sum_{j \in J} y_{ij} + y_{m+1,\,n+1} = \sum_{j \in J} (B_j - \sum_{i \in I} x_{ij})$$

$$= \sum_{j \in J} B_j - \sum_{i \in I} \sum_{j \in J} x_{ij}$$

$$= \sum_{j \in J} B_j - P$$

$$= A'_{m+1}$$

$$\Rightarrow \sum_{j \in J'} y_{ij} = A_i^{'} \ ; \ \forall i \in I'$$

Similarly, it can be shown that $\sum_{i \in I'} y_{ij} = B_j^{'} \ ; \ \forall j \in J'$

Therefore, $\{y_{ij}\}_{I' \times J'}$ is a cfs to (P2′).

Conversely, let $\{y_{ij}\}_{I' \times J'}$ be a cfs to (P2′) .Define x_{ij} , $i \in I$, $j \in J$ by the following transformation.

$x_{ij} = y_{ij}$, $i \in I$, $j \in J$

It implies that $l_{ij} \le x_{ij} \le u_{ij}$, $i \in I$, $j \in J$

Now for $i \in I$, the source constraints in (P2′) implies

$$\sum_{j \in J'} y_{ij} = A_i^{'} = A_i$$

$$\sum_{j \in J} y_{ij} + y_{i,n+1} = A_i$$

$$\Rightarrow a_i \le \sum_{j \in J} y_{ij} \le A_i \qquad (\text{since } 0 \le y_{i,n+1} \le A_i - a_i \quad , i \in I)$$

Hence, $a_i \le \sum_{j \in J} x_{ij} \le A_i \quad , i \in I$

Similarly, for $j \in J$, $b_j \le \sum_{i \in I} x_{ij} \le B_j$

For i= m+1,

$$\sum_{j \in J'} y_{m+1,j} = A_{m+1}^{'} = \sum_{j \in J} B_j - P$$

$$\Rightarrow \sum_{j \in J} y_{m+1,j} = \sum_{j \in J} B_j - P \qquad (\text{because } y_{m+1,n+1} = 0)$$

Now, for $j \in J$ the destination constraints in (P2′) give

$$\sum_{i \in I} y_{ij} + y_{m+1,j} = B_j$$

Therefore, $\sum_{i \in I} \sum_{j \in J} y_{ij} + \sum_{j \in J} y_{m+1,j} = \sum_{j \in J} B_j$

$$\sum_{i \in I} \sum_{j \in J} y_{ij} = \sum_{j \in J} B_j - \sum_{j \in J} y_{m+1,j} = P$$

$$\Rightarrow \sum_{i \in I} \sum_{j \in J} x_{ij} = P$$

Therefore $\{x_{ij}\}_{I \times J}$ is a feasible solution to (P2)

Remark 1: If (P2′) has a cfs ,then since $c'_{m+1,n+1}$=M and $d'_{m+1,n+1}$= M, it follows that non corner feasible solution can not be an optimal solution of (P2′) .

Lemma 2: The value of the objective function of problem (P2) at a feasible solution $\{x_{ij}\}_{I \times J}$ is equal to the value of the objective function of (P2´) at its corresponding cfs $\{y_{ij}\}_{I' x J'}$ and conversely.

Proof: The value of the objective function of problem (P2´) at a feasible solution $\{y_{ij}\}_{I' x J'}$ is

$$z = \left[\frac{\sum_{i \in I'} \sum_{j \in J'} c'_{ij} y_{ij}}{\sum_{i \in I'} \sum_{j \in J'} d'_{ij} y_{ij}} \right]$$

$$= \left[\frac{\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}}{\sum_{i \in I} \sum_{j \in J} d_{ij} x_{ij}} \right] \text{ because } \left\{ \begin{array}{l} c'_{ij} = c_{ij}, \forall i \in I, j \in J \\ d'_{ij} = d_{ij}, \forall i \in I, j \in J \\ x_{ij} = y_{ij}, \forall i \in I, j \in J \\ c'_{i,n+1} = c'_{m+1,j} = 0; \forall i \in I, j \in J \\ d'_{i,n+1} = d'_{m+1,j} = 0; \forall i \in I, j \in J \\ y_{m+1,n+1} = 0 \end{array} \right\}$$

= the value of the objective function of (P2) at the corresponding feasible solution $\{x_{ij}\}_{I x J}$

The converse can be proved in a similar way.

Lemma 3: There is a one –to-one correspondence between the optimal solution to (P2) and optimal solution to the corner feasible solution to (P2´) .

Proof: Let $\{\overset{\circ}{x}_{ij}\}_{I \times J}$ be an optimal solution to (P2) yielding objective function value z^0 and $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ be the corresponding cfs to (P2´).Then by Lemma 2, the value yielded by $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ is z^0 .If possible, let $\{\overset{\circ}{y}_{ij}\}_{I' \times J'}$ be not an optimal solution to (P2´) . Therefore, there exists a cfs $\{y'_{ij}\}$ say ,to(P2´) with the value $z^1 < z^0$. Let $\{x'_{ij}\}$ be the corresponding feasible solution to (P2).Then by lemma 2,

$$z^1 = \left[\frac{\sum_{i\in I}\sum_{j\in J} c_{ij}x'_{ij}}{\sum_{i\in I}\sum_{j\in J} d_{ij}x'_{ij}}\right]$$, a contradiction to the assumption that $\{\overset{\circ}{x}_{ij}\}_{I\times J}$ is an optimal solution of (P2).Similarly, an optimal corner feasible solution to (P2´) will give an optimal solution to (P2).

Theorem 3: Optimizing (P2´) is equivalent to optimizing (P2) provided (P2) has a feasible solution.

Proof: As (P2) has a feasible solution , by lemma 1,there exists a cfs to (P2´) .Thus by remark 1,an optimal solution to (P2´) will be a cfs. Hence , by lemma 3,an optimal solution to (P2) can be obtained.

4 Algorithm:

Step 1 : Given a fractional capacitated transportation problem (P1),separate the problem (P1) in to two problems (P2) and (P3).Form the related transportation problems (P2´) and (P3´).Find a basic feasible solution of problem (P2´) with respect to the variable cost only. Let B be its corresponding basis.

Step 2: Calculate θ_{ij} , $u_i^1, u_i^2, v_j^1, v_j^2, z_{ij}^1, z_{ij}^2; i \in I, j \in J$ such that

$$u_i^1 + v_j^1 = c_{ij} \quad \forall (i, j) \in B$$

$$u_i^2 + v_j^2 = d_{ij} \quad \forall (i, j) \in B$$

$$u_i^1 + v_j^1 = z_{ij}^1 \quad \forall (i, j) \in N_1 \text{ and } N_2$$

$$u_i^2 + v_j^2 = z_{ij}^2 \quad \forall (i, j) \in N_1 \text{ and } N_2$$

θ_{ij} = level at which a non basic cell (i,j) enters the basis replacing some basic cell of B.

N_1 and N_2 denotes the set of non basic cells (i,j) which are at their lower bounds and upper bounds respectively. $u_i^1, v_j^1, u_i^2, v_j^2$ are the dual variables which are determined by using the above equations and taking one of the u_i's or v_j's as zero.

Step3(a): Calculate N°, D° where $N^\circ = \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij}, D^\circ = \sum_{i\in I}\sum_{j\in J} d_{ij}x_{ij}$

Step3(b): Calculate A_{ij}^1 and A_{ij}^2 where $A_{ij}^1 = \theta_{ij}(c_{ij} - z_{ij}^1); \forall (i,j) \notin B$ and $A_{ij}^2 = \theta_{ij}(d_{ij} - z_{ij}^2); \forall (i,j) \notin B$.

Step 4(a): Find $\Delta_{ij} = D^0(c_{ij} - z_{ij}^1) - N^0(d_{ij} - z_{ij}^2); \forall (i,j) \notin B$

Step 4(b): Find $\delta_{ij}^1 ; \forall (i,j) \in N_1$ and $\delta_{ij}^2 ; \forall (i,j) \in N_2$ where

$$\delta_{ij}^1 = \left[\frac{\theta_{ij}\Delta_{ij}}{D^\circ\left[D^\circ + A_{ij}^2\right]}\right]; \forall (i,j) \in N_1 \text{ and}$$

$$\delta_{ij}^2 = \left[-\frac{\theta_{ij}\Delta_{ij}}{D^\circ\left[D^\circ - A_{ij}^2\right]}\right]; \forall (i,j) \in N_2$$ where N_1 and N_2 denotes the set of non basic cells (i,j) which are at their lower bounds and upper bounds respectively.

If $\delta_{ij}^1 \geq 0; \forall (i,j) \in N_1$ and $\delta_{ij}^2 \geq 0; \forall (i,j) \in N_2$ then the current solution so obtained is the optimal solution to (P2´) and subsequently to (P2).Then go to step (5). Otherwise some $(i,j) \in N_1$ for which $\delta_{ij}^1 < 0$ or some $(i,j) \in N_2$ for which $\delta_{ij}^2 < 0$ will enter the basis. Go to step 2.

Step 5: Let Z^1 be the optimal cost of (P2´) yielded by the basic feasible solution $\{y'_{ij}\}$. Find all alternate solutions to the problem (P2´) with the same value of the objective function. Let these solutions be $X_1, X_2, \ldots X_n$ and $T^1 = \min_{X_1, X_2 \ldots X_n} \max_{i\in I, j\in J'} t_{ij} / x_{ij} > 0$ .Then the corresponding pair (Z^1, T^1) will be the first time cost trade off pair for the problem (P1).To find the second cost- time trade off pair, go to step 6.

Step6: Define $c_{ij}^1 = \begin{cases} M & \text{if } t_{ij} \geq T^1 \\ c_{ij} & \text{if } t_{ij} < T^1 \end{cases}$

where M is a sufficiently large positive number. Form the corresponding capacitated fixed charge quadratic transportation problem with variable cost c_{ij}^1.Repeat the above process till the problem becomes infeasible. The complete set of cost- time trade off pairs of (P1) at the end of q^{th} iteration is given by $(Z^1, T^1), (Z^2, T^2) \ldots\ldots (Z^q, T^q)$ where $Z^1 \leq Z^2 \leq \ldots \leq Z^q$ and $T^1 > T^2 > \ldots > T^q$.

Remark 2: The pair (Z^1, T^q) with minimum cost and minimum time is the ideal pair which can not be achieved in practice except in some trivial case.

Convergence of the algorithm: The algorithm will converge after a finite number of steps because we are moving from one extreme point to another extreme point and the problem becomes infeasible after a finite number of steps.

5 Numerical Illustration:

Consider a 3 x 3 fractional capacitated transportation problem with restricted flow .Table 1 gives the values of c_{ij}, d_{ij}, A_i ,B_j for i=1,2,3 and j=1,2,3

Table 1: cost matrix of problem (P2)

	D_1	D_2	D_3	A_i
O_1	5	9	9	30
O_2	4 3	6 7	2 4	40
O_3	2 2	1 9	1 4	50
B_j	30	20	30	

Note: values in the upper left corners are c_{ij} 's and values in lower left corners are d_{ij} 's for i=1,2,3.and j=1,2,3.

Also, $3 \le \sum_{j=1}^{3} x_{1j} \le 30$, $10 \le \sum_{j=1}^{3} x_{2j} \le 40$, $10 \le \sum_{j=1}^{3} x_{3j} \le 50$, $5 \le \sum_{i=1}^{3} x_{i1} \le 30$,

$$5 \le \sum_{i=1}^{3} x_{i2} \le 20, \quad 5 \le \sum_{i=1}^{3} x_{i3} \le 30$$

$1 \le x_{11} \le 10$, $2 \le x_{12} \le 10$, $0 \le x_{13} \le 5$,$0 \le x_{21} \le 15$, $3 \le x_{22} \le 15$, $1 \le x_{23} \le 20$, $0 \le x_{31} \le 20$, $0 \le x_{32} \le 13$, $0 \le x_{33} \le 25$

Table 2 gives the values of t_{ij}'s for i=1,2,3 and j=1,2,3

Table 2 : Time matrix of problem (P3)

	D_1	D_2	D_3
O_1	15	8	13
O_2	10	13	14
O_3	12	10	9

Let the restricted flow be P = 40 where P = 40 < min $\left(\sum_{i=1}^{3} A_i = 120, \sum_{j=1}^{3} B_j = 80 \right)$

Introduce a dummy origin and a dummy destination in Table 1 with $c_{i4} = 0 = d_{i4}$ for all i = 1,2 ,3 and $c_{4j} = 0 = d_{4j}$for all j = 1,2,3 . $c_{44}=d_{44}=M$ where M is a large positive number. Also we have $0 \le x_{14} \le 27$, $0 \le x_{24} \le 30$, $0 \le x_{34} \le 40$, $0 \le x_{41} \le 25$, $0 \le x_{42} \le 15$, $0 \le x_{43} \le 25$ and $F_{4j} = 0$ for j=1,2,3,4 In this way , we form the problem (P2´).Similarly on introducing a dummy origin and a dummy destination in Table 2 with $t_{i4} = 0$ for i=1,2 ,3and $t_{4j}= 0$ for j=1,2,3,

$t_{44} > \max \; t_{ij} / x_{ij} > 0 \;\; \forall i \in I, j \in J$,we form problem (P3´) . Also, $B_4 = \sum_{i=1}^{3} A_i - P$ =120-40 = 80

and $A_4 = \sum_{j=1}^{3} B_j - P$ = 80-40 = 40

Now we find an initial basic feasible solution of problem (P2´) which is given in table 3 below.

Table 3: A basic feasible solution of problem (P2´)

	D_1	D_2	D_3	D_4	u_i^1	u_i^2
O_1	5 $\underline{1}$ 4	9 $\underline{2}$ 2	9 $\underline{0}$ 1	0 **27** 0	0	0
O_2	4 $\underline{0}$ 3	6 $\underline{3}$ 7	2 **7** 4	0 $\overline{30}$ 0	1	0

O_3	2 2 **4**	1 9 $\overline{13}$	1 4 **10**	0 0 **23**	0	0
O_4	0 0 $\boxed{\overline{25}}$	0 0 **2**	0 0 **13**	M M	-1	-4
v_j^1	2	1	1	0		
v_j^2	2	4	4	0		

Note: entries of the form $\underline{a}$ and $\overline{b}$ represent non basic cells which are at their lower and upper bounds respectively. Entries in bold are basic cells.

$N^0 = 86$, $D^0 = 222$

Table 4: Calculation of δ_{ij}^1 and δ_{ij}^2

NB	O_1D_1	O_1D_2	O_1D_3	O_2D_1	O_2D_2	O_2D_4	O_3D_2	O_4D_1
θ_{ij}	4	2	5	4	2	10	13	10
$c_{ij} - z_{ij}^1$	3	8	8	1	4	-1	0	-1
$d_{ij} - z_{ij}^2$	2	-2	-3	1	3	0	5	2
A_{ij}^1	12	16	40	4	8	-10	0	-10
A_{ij}^2	8	-4	-15	4	6	0	65	20
Δ_{ij}	494	1948	2034	136	630	-222	-430	-394
δ_{ij}^1	0.0387	0.0805	0.2213	0.0108	0.02489			
δ_{ij}^2						0.04504	0.16038	0.08786

Since $\delta_{ij}^1 \geq 0; \forall (i,j) \in N_1$ and $\delta_{ij}^2 \geq 0; \forall (i,j) \in N_2$, the solution given in table 3 is an optimal solution to problem (P2´).Therefore $Z^1 = 0.3873$ and corresponding time is $T^1 = 15$. Hence the first cost time trade off pair is (0.3873,15).

Define $c_{ij}^1 = \begin{Bmatrix} M & \text{if } t_{ij} \geq T^1 = 15 \\ c_{ij} & \text{if } t_{ij} < T^1 = 15 \end{Bmatrix}$ and solving the resulting problem , the next trade off pair is (0.3873,14)

$$\text{Define } c_{ij}^{2} = \begin{cases} M & \text{if } t_{ij} \geq T^{2} = 14 \\ c_{ij} & \text{if } t_{ij} < T^{2} = 14 \end{cases}$$

A basic feasible solution to the new cost problem is given in table 5 below.

Table 5: A basic feasible solution to the new cost problem

	D_1	D_2	D_3	D_4	u_i^1	u_i^2
O_1	M $\underline{1}$ M	9 $\underline{2}$ 2	9 1	0 **27** 0	1	4
O_2	4 **6** 3	6 $\underline{3}$ 7	M $\underline{1}$ M	0 $\overline{30}$ 0	4	3
O_3	2 2	1 $\overline{13}$ 9	1 **14** 4	0 **23** 0	1	4
O_4	0 $\boxed{}$ **23** 0	0 **2** 0	0 **15** 0	M M	0	0
v_j^1	0	0	0	-1		
v_j^2	0	0	0	-4		

$N^0 = 94$, $D^0 = 224$, $Z = 0.4196$

Table 6: Calculation of δ_{ij}^1 and δ_{ij}^2

NB	O_1D_2	O_1D_3	O_2D_2	O_2D_4	O_3D_1	O_3D_2
θ_{ij}	2	5	2	9	10	11
$c_{ij} - z_{ij}^1$	8	8	2	-3	1	0
$d_{ij} - z_{ij}^2$	-2	-3	4	1	-2	5
A_{ij}^1	16	40	4	-27	10	0
A_{ij}^2	-4	-15	8	9	-20	55
Δ_{ij}	1980	2074	72	-766	412	-470
δ_{ij}^1	0.8035	0.2215	0.0027		0.0902	

δ_{ij}^{2}				0.1431		0.1365

Since $\delta_{ij}^{1} \geq 0; \forall (i,j) \in N_1$ and $\delta_{ij}^{2} \geq 0; \forall (i,j) \in N_2$, the solution given in table 5 is an optimal solution to problem (P2´).Therefore Z^3=0.4196 and corresponding time is T^3 = 13 . Hence the third time cost trade off pair is (0.4196,13).

Define $c_{ij}^{3} = \begin{Bmatrix} M & \text{if } t_{ij} \geq T^3 = 13 \\ c_{ij} & \text{if } t_{ij} < T^3 = 13 \end{Bmatrix}$

and solving the resulting problem , the next trade off pair is (0.4196,10)

Define $c_{ij}^{4} = \begin{Bmatrix} M & \text{if } t_{ij} \geq 10 \\ c_{ij} & \text{if } t_{ij} < 10 \end{Bmatrix}$

and on solving, the problem becomes infeasible. Hence the cost time trade off pairs are (0.3873,15) , (0.3873,14),(0.4196 , 13), (0.4196 , 10).

Conclusion:

In order to solve a capacitated fixed charge bi-criterion fractional transportation problem with restricted flow, given problem is separated in to two problems. A related transportation problem is formulated and the efficient cost- time trade off pairs to the given problem are shown to be derivable from this related transportation problem. After calculating cost, corresponding time is read. This is the first time cost trade off pair. Proceeding like this, we get the various trade off pairs.

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